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THEORETICAL ASPECTS OF PLANAR SOUND PROPAGATION IN THE ATMOSPHERE

By

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GEORGE C. MARSHALL SPACE FLIGHT CENTER

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ABSTRACT

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The courses of sound rays in the atmosphere, depending as they sometimes are on rather erratic temperature and wind distributions, may make a pattern so intricate as to virtually preclude a mathematical treatment. The problem becomes better manageable with a more orderly distribution of meteorological parameters such that, in a wedge-like slice of the atmosphere bounded by two adjacent half-planes through the source vertical, the state of the gas mixture can be supposed to be steady and independent of the angle variation in between the planes. The wedge-like structure can then be replaced by a representative half-plane, so that the problem is reduced to two dimensions, both spatial.

The conventional approach assumes that in such a half-plane the observed distribution of the propagation velocity can be approximated by a piecewise linear variation with height alone, the simplest case being where it is linear without break (one-layered atmosphere). It cannot cope with situations involving a velocity variation with horizontal distance, as, e.g., may be found over hilly terrain or around the seashore. The bulk of the present report is devoted to developing theoretical means for computing planar sound ray patterns that evolve from more general velocity distributions and, conversely, for finding velocity fields that correspond to a given ray pattern. Several examples of such interrelationships are given and discussed in some detail. In all these cases the atmosphere is one-layered in the wider sense that the velocity field can be described by a single differentiable function of height and distance. Further work will have to establish in what way such theoretical fields can be used to approximate actual distributions known from observation or from suitable data extrapolation.

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TECHNICAL AND SCIENTIFIC STAFF
AEROBALLISTICS DIVISION

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SUMMARY

The state of the atmosphere can be considered as aerodynamically steady with reference to a point source of sound that is embedded in it at a fixed location (static source). The distribution of temperature and wind may be such that the sound rays emitted under various angles of departure in a vertical half-plane in essence won't leave that plane during their entire course. The fundamental, then two-dimensional, acoustical eikonal equation becomes amenable to analytic treatment if (a) the wind speed is horizontal and decidely inferior to the thermodynamic sound speed, and (b) if sound rays of small elevation angles are considered only (as it usually suffices to do in practice). The eikonal equation, simplified by force of these assumptions, can be converted into a differential equation for the ray slope; the latter is solved in general for arbitratily prescribed distributions of the propagation velocity in the half-plane selected. It is shown that identically the same ray pattern can issue from a multitude of such distributions. An inverse method which starts out with a prescribed ray field and determines velocity fields that would produce it is also developed. Both methods can be used for a systematic survey of the interdependence of planar velocity and ray fields, a first step toward practical application of the theory. More such relationships can be obtained if the system of coordinates is rotated about the sound source. A number of examples employing these methods are given and discussed in more or less detail. A last section touches on the general question of focal point formation.

SECTION I. INTRODUCTION

The shell of acoustical energy loosened in an element of time from a point source of sound can be broken up into infinitely many infinitesimally small fragments associated with the infinitely many directions of departure from the source. The course followed by any such minute parcel is defined as a sound ray. The longitudinal vibration contained in a parcel or pulse is not necessarily in the direction of the ray; it is so in the absence of wind only. Similarly the keel of a boat is not in the direction of its movement, seen from ashore, unless it moves in standing water.

The space around the source can be regarded as filled with a family of surfaces (wave fronts) on which the phase of the passing sound vibration is the same everywhere while of course it changes with time. These fronts, as their name would suggest, may also be viewed as the successive locations of the forward phase of all the pulses that have left the source at the same time, t_o. On arriving at such a surface, their direction of motion, i.e., the direction of the sound rays, can make any angle with the surface depending on wind direction and magnitude.

Atmospheric conditions can be extremely variegated. They will not only change from spot to spot, but, at the same location, with time as well. If we restrict the considerations to a static source, the temporal variation with respect to it will, as a rule, be so slow that it can be disregarded in view of the relatively short activity of the source. The then locally constant thermodynamic sound speed, i.e., the propagation velocity in a windless atmosphere, varies with the local temperature and the molecular weight of the carrier gas. If the atmosphere is considered as a single gas of constant composition, this speed is given by

$$c^{(o)} = \sqrt{\gamma \frac{R_{univ}}{\mu} T^{(o)}} \; . \tag{1}$$

With the usual near-ground composition

$$c^{(o)} \approx 20 \sqrt{T^{(o)}}$$

where $c^{(o)}$ emerges in meters/sec. when the absolute temperature is measured in degrees Kelvin.

If in such a single-gas, steady-state atmosphere the square of the wind velocity is negligibly small when compared to the square of the thermodynamic sound speed, a significant simplification can be made in that the velocity vector of energy propagation is very nearly identical with the vector of the phase velocity, $\underline{V}_f$, normal to the fronts and can be replaced by it [1]. The sound rays then can be calculated as the fronts' orthogonal trajectories which, in themselves of little practical interest, can now be thought of as the traveling routes of energy. The investigations to follow are based on this concept.

All figures have been computed by James Mabry and have been drawn up by H. W. Vardaman.

SECTION II. THE TWO-DIMENSIONAL EIKONAL EQUATION

An access to the problem is opened by the work of Blokhintzev [2]. In his chapter on "ray" acoustics (implying that diffraction processes are disregarded), he shows that if the argument of the sinusoidal, small-amplitude sound oscillation is written as $\omega t - k_o W$, the phase factor W will obey the acoustical eikonal equation

$$|\text{grad } W|^2 = \left(\frac{q}{c^{(o)}}\right)^2 \tag{2}$$

where

$$q = c_o - \underline{w}^{(o)} \cdot \text{grad } W \tag{3}$$

c_o = arbitrary referency velocity

$\underline{w}^{(o)}$ = vector of wind velocity.

Equation (2) is approximative for sufficiently large values of the wave number k_o.

By definition, the gradient of the phase factor W points into the direction, $\underline{n}$, of the wave front normals, so that

$$\text{grad } W = \underline{n} \, |\text{grad } W|.$$

The relations (2) and (3) then combine to give

$$|\text{grad } W| = \frac{c_o}{V_f} \tag{4}$$

where

$$V_f = c^{(o)} + \underline{n} \cdot \underline{w}^{(o)} \tag{5}$$

is the absolute value of the wave velocity [1]. In general, the quantities V_f and W will depend on all three space variables, i.e., in a cylindrical system, on

x (standing for the customary r)

y (height)

φ (azimuth).

Let us take the y-axis as the ground vertical at the sound source. With a favorable distribution of meteorological parameters, the dependence on φ can be disregarded.

Consider two half-planes joined along the y-axis and separated by the small angle $\Delta\varphi$. If at any two points in these planes, which are at the same height and equally distant from the source, the differences in meteorological data are of order $(x\Delta\varphi)^2$ rather than of order $(x\Delta\varphi)$, and if this is true for all the half-planes in between, then the data will depend on x and y alone (in the first approximation); the wedge-like spatial sector can be taken as part of a substitute atmosphere where this is universally true and where therefore the wave fronts are symmetric about the y-axis.* In these circumstances, equation (4) can be reduced to two dimensions, since the dependency on the azimuth can be dropped. It may be written as

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = \left(\frac{c_o}{V_f}\right)^2. \tag{6}$$

The sound rays, in general of double curvature, are now confined to a representative half-plane in which the sound propagation is being investigated. The variable x, originally the radius-vector in a cylindrical system, is essentially positive. The ordinate y should be permitted to take on negative values as well, since mathematically the half-plane is not bounded in the downward direction and physically the sound source could be located above ground level.

The surfaces of constant phase appear as curved lines in the half-plane considered. Their normals are tangents to the sound rays [if $w^{(o)} \ll c^{(o)}$] and can therefore be written as

$$\underline{n} = i_1 \cos\theta + i_2 \sin\theta$$

where θ is the local angle of ray elevation. The component in the azimuthal direction (i_3) is zero. If u_1, u_2, u_3 are the components of the wind vector $\underline{w}^{(o)}$ in the three spatial directions, it follows that

$$\underline{n} \cdot \underline{w}^{(o)} = u_1 \cos\theta + u_2 \sin\theta.$$

The component u_3 has no bearing on the two-dimensional problem, since $\underline{n} \cdot \underline{w}^{(o)}$ is the projection of $\underline{w}^{(o)}$ on the direction $\underline{n}$ and is therefore not affected by the azimuthal component which is normal to $\underline{n}$.

*This condition prevails if, for example, temperature and wind vary with height alone, or nearly do so.

The vertical component, u_2, is hard to observe and is often quite small so that it is taken as zero by many authors, e.g., in references 1, 2, 3, 4, 5. This simplification will be adopted in the present paper. (As a consequence, the treatment is not applicable where strong vertical currents are suspected or shown to exist.) Furthermore, in accordance with references 1, 3, and 4, only low-lying rays with small angles of elevation will be considered so that cos θ remains close to unity. This further simplification is often permissible in practical applications. In view of the earlier restriction requiring the wind speed to be much smaller than the sound speed, relation (5) may then be written as

$$V_f = c^{(o)} + u_1.$$

The acoustical propagation speed appears as the algebraic sum of the thermodynamic speed and the component of the horizontal wind vector in the selected half-plane. A very considerable mathematical advantage is gained with this: The right side of equation (6) can now be considered a given function of x and y while so far it had depended on the unknown quantity $\underline{n}$.

SECTION III. PLANAR SOUND RAYS AND WAVE FRONTS IN GENERAL

The solution sought to equation (6) does not have to satisfy ordinary initial value conditions. The aim is not to determine a particular integral surface that, on a certain (non-characteristic) curve prescribed in it, should be sloping in a prescribed way. The situation is quite different. As evidenced by the presence of the arbitrary reference velocity c_o in equation (6), the W-values themselves are not essential in the problem; neither are the values of the first partial derivatives. Rather, since W = const. is the equation of the curves of constant phase, the interest centers about the quotient of the derivatives, as

$$\frac{\partial W}{\partial x}\,dx + \frac{\partial W}{\partial y}\,dy = 0 \tag{7}$$

is the differential equation of these curves. That of the sound rays may be written as

$$\frac{dy}{dx} = \frac{W_y}{W_x} \tag{8}$$

where the subscripts y and x indicate partial derivation with respect to these variables.

The sound source, which will be located at the origin, is a singular point since all the rays take their departure from it. In terms of x and y the quotient on the right side of equation (8) must therefore be indeterminate at the origin. This cannot be accomplished by setting $W_y = W_x = 0$ at the origin. Such a condition could not be satisfied by any solution of the differential equation (6), since its right side is a finite quantity unless the propagation velocity at the sound source

$$V_o = V_f(0, 0)$$

is infinite, which it is not. The only other alternative is to require one of the partial derivatives to be indeterminate. The other one then is necessarily also indeterminate, since the sum of their squares is to have the determinate value

$$\left(\frac{c_o}{V_o}\right)^2 .$$

The problem on hand can then be formulated as follows: Find a solution to equation (6) such that the partial derivative W_x (or W_y) remains indeterminate at the sound source. It is clear that with this single stipulation the solution cannot be unique. Uniqueness is required only as far as the ratio of the first-order partial derivatives is concerned which, by equation (8), defines the slope of the sound rays. If it is viewed as an equation for W, no further requirements as to initial values are inherent in the problem. However, it will be expected that the solution properly furnishes curves of constant phase and the sound rays. The above stipulation was prompted by a physical condition imposed by the latter, while the wave fronts are merely the orthogonal trajectories of the rays.

Such a solution has been given in reference 1 for the velocity distribution

$$V_f = V_o + \mu y \tag{9}$$

where μ (either > 0 or < 0) is the constant gradient of V_f in the y-direction. It is the purpose of the present paper to attack the problem when more general distributions are prescribed.

From the earlier work, some advantage can be foreseen when dimensionless quantities are introduced by the transformations

$$\begin{cases} \eta = 1 + \dfrac{\mu}{V_o} y = 1 + \lambda y \\ \xi = \lambda x \\ \emptyset = \lambda W \\ v = \dfrac{V_f}{V_o} . \end{cases} \tag{10}$$

In a general case, μ will be related to the y-component of grad V_f, taken at the source. (In the special field (9) the constant μ is the gradient of V_f everywhere.) It follows that the constant λ may have a positive or a negative value, making ξ either always positive or always negative (for $x > 0$). For the distribution (9), $v = \eta$.

Relations (10) transform equation (6) into

$$\left(\frac{\partial \phi}{\partial \xi}\right)^2 + \left(\frac{\partial \phi}{\partial \eta}\right)^2 = \frac{1}{v^2} , \tag{11}$$

if c_o is put equal to V_o, which we are free to do. The method employed in reference 1 for obtaining the solution to equation (11) when $v = \eta$ does not lend itself to easy generalization. A more systematic manner of handling the problem would make use of the characteristic equations associated with partial differential relationships. The dimensionless velocity can then be prescribed as an arbitrary positive function of ξ and η, restricted only by the condition that $v = 1$ at the sound source which, if located at the origin of the (x, y)-system, by transformation (10) has the dimensionless coordinates $\xi = 0$, $\eta = 1$.

There is no real interest in determining the function ϕ itself. It appears more appropriate to deal with a differential equation in which, e.g., the slope, s, of the sound rays is the unknown function. Such an equation can be developed from equation (11); contrary to (11), it will be quasi-linear, which makes it even more attractive.

The slope $s = \frac{dy}{dx}$ obeys the equation (8). If subjected to the transformation (10) it reads

$$s = \frac{d\eta}{d\xi} = \frac{\phi_\eta}{\phi_\xi} .$$

With the aid of the differential equation (11) both ϕ_ξ and ϕ_η can be expressed in terms of s:

$$\phi_\xi = \pm \frac{1}{v} \frac{1}{\sqrt{s^2 + 1}}$$

$$\phi_\eta = \pm \frac{1}{v} \sqrt{\frac{s^2}{s^2 + 1}}$$

where the quantity v is positive and the square roots are taken as absolute.

The condition $\phi_{\xi\eta} = \phi_{\eta\xi}$ requires that

$$\pm \frac{\partial}{\partial\eta} \frac{1}{v} \frac{1}{\sqrt{s^2 + 1}} = \pm \frac{\partial}{\partial\xi} \frac{1}{v} \sqrt{\frac{s^2}{s^2 + 1}} .$$

The double sign at the left refers to the signs of ϕ_ξ, that at the right to those of ϕ_η. If both ϕ_ξ and ϕ_η are positive or both negative, s will be a positive quantity; otherwise, s will be negative. It follows that it is advisable to affix a sign to the square root of s^2 in putting

$$\sqrt{s^2} = s$$

rather than

$$\sqrt{s^2} = |s|$$

since the double signs then cancel out. With this provision the cross-derivative relation goes into

$$s_\xi + s s_\eta + (s^2 + 1) \left(\frac{v_\eta}{v} - s \frac{v_\xi}{v}\right) = 0. \tag{12}$$

In terms of a parameter, ρ, the associated characteristic equations for the three independent variables ξ, η, s may be written as

$$\begin{cases} \dfrac{d\xi}{d\rho} = v \\[2ex] \dfrac{d\eta}{d\rho} = vs \\[2ex] \dfrac{ds}{d\rho} = (s^2 + 1)\ (s\ v_\xi - v_\eta). \end{cases} \tag{13}$$

Any integral of the system (13) is also an integral of equation (12). Moreover, since (12) is quasi-linear, any two integrals ψ_1 (ξ, η, s) and ψ_2 (ξ, η, s) of the system give rise to a quite general integral

$$\psi = \Omega\ (\psi_1, \psi_2)$$

where Ω is an arbitrary differentiable function. This theorem is important in our context; it will enable us to find an integral such that s remains indeterminate at $\xi = 0$, $\eta = 1$. It will be shown that, if the relationship

Ω = const. of ψ_1 and ψ_2 is written as

$$\psi_1 = G(\psi_2) \tag{14}$$

the still arbitrary function G can be specialized in such a way that the above condition is satisfied. Consider the values of the known integrals ψ_1 and ψ_2 at the sound source:

$$\tilde{\psi}_1 = \psi_1(0,\ 1,\ s).$$

$$\tilde{\psi}_2 = \psi_2(0,\ 1,\ s).$$

Here the variable s can be eliminated to give

$$\tilde{\psi}_1 = H(\tilde{\psi}_2) = f(s) \tag{15}$$

where H is a known function. In putting

$$G \equiv H$$

two things will be accomplished. On the one hand, it will be impossible to compute the value s at $\xi = 0$, $\eta = 1$, since equation (14) will degenerate into the identity f = f; hence, s remains indeterminate at the sound source. On the other hand, the solution will be unique, as there is only one function H that relates ψ_2 to ψ_1 at the sound source.* Thus, the general problem is solved in principle. For any prescribed v-distribution, it is reduced to finding two integrals of the system (13). To be sure,

---

*If a pair of different integrals, ψ_1' and ψ_2', give rise to the relation

(a) $\psi_1' = H'(\psi_2')$

we can introduce $\psi_1' = \Omega_1(\psi_1,\ \psi_2)$, $\psi_2' = \Omega_2(\psi_1,\ \psi_2)$ and solve the ensuing equation for ψ_1:

(b) $\psi_1 = H^*(\psi_2)$.

If equation (a) reduces to an identity at the source, equation (b) must do likewise, so that $H^* = H$. The function $s' = s'(\xi,\ \eta)$ that becomes indeterminate at the source by virtue of equation (a) will be carried through the Ω-transformations into equation (b) and then must be identical with the function $s = s(\xi,\ \eta)$ which would issue from $\psi_1 = H(\psi_2)$.

those integrals must be independent of each other lest relation (15) become an identity, too. In other words, the matrix

$$\begin{bmatrix} \frac{\partial\psi_1}{\partial\xi} & \frac{\partial\psi_1}{\partial\eta} & \frac{\partial\psi_1}{\partial s} \\ \frac{\partial\psi_2}{\partial\xi} & \frac{\partial\psi_2}{\partial\eta} & \frac{\partial\psi_2}{\partial s} \end{bmatrix}$$

must be of rank 2 so that at least one of the two-rowed determinants contained in it is not identically zero in the domain where the ψ's are defined. If two such integrals are known, the solution $s = g(\xi, \eta)$ can be obtained from equation (14) using G = H (although this may meet with algebraic difficulties). The sound rays and wave fronts finally are described by the differential equations

$$\frac{d\eta}{d\xi} = s \quad \text{and} \quad \frac{d\eta}{d\xi} = -\frac{1}{s}\,. \tag{16}$$

The integration constant, C^*, of the former is to define an individual sound ray which, however, grows into a separate entity only after it has left the source. There is no distinction between rays at the sound source itself so that in the solution the integration constant will always be tied to a linear function that vanishes at $\xi = 0$, $\eta = 1$. Its derivative, must not and does not vanish there; for if the solution is differentiated with respect to ξ, it must be feasible to determine C^* from the equation

$$\frac{d\eta}{d\xi} = g(\xi, \eta, C^*) \tag{17}$$

in putting

$$g(0, 1, C^*) = \tan\theta_o \tag{18}$$

since the initial angle, θ_o, of elevation is the discerning element for rays.

It may seem unnecessary to also obtain the integral, $X(\xi, \eta)$, of the second equation (16). After all, the immediate interest is in the sound ray pattern as prompted by a prescribed velocity distribution $v(\xi, \eta)$. However, a curious and far-reaching application can be made of the wave front integral.

Since $\chi(\xi, \eta) = C$ is the equation of the family of wave fronts, their slopes may be written as

$$\frac{d\eta}{d\xi} = -\frac{1}{s} = -\frac{\chi_\xi}{\chi_\eta}$$

so that

$$\chi_\eta - s\chi_\xi = 0, \tag{19}$$

where s is the sound ray slope as obtained from the basic differential equation (12) on the strength of a given velocity distribution $v(\xi, \eta)$.

Consider now the velocity distribution

$$\tilde{v} = v\ F(\chi) \tag{20}$$

where F is any positive differentiable function of χ restricted only to $F = 1$ at the source. Let F' denote its derivative. The sound ray slope, $\tilde{s}$, pertaining to $\tilde{v}$ obeys the equation (12) which may be written as

$$\tilde{s}_\xi + \tilde{s}\tilde{s}_\eta + (\tilde{s}^2 + 1)\left(\frac{v_\eta}{v} - \tilde{s}\,\frac{v_\xi}{v}\right) = -\frac{F'}{F}\,(\chi_\eta - \tilde{s}\chi_\xi)\ (\tilde{s}^2 + 1).$$

The left side here is zero if one puts $\tilde{s} = s$; so is the right side according to relation (19). Hence, $\tilde{s} = s$ is the required solution, since s is indeterminate at the source. In view of the equations (16), the distributions v and $\tilde{v}$ then result in the same families of sound rays and wave fronts, the only difference being that the rays and fronts move at a different rate. If the solution is found for a certain distribution v, it is also found for infinitely many other distributions, $\tilde{v} = v\ F(\chi)$, since F is a quite arbitrary function.

This remarkable fact can be demonstrated in a heuristic manner. Consider two wave fronts, C and $\tilde{C}$, as given by the two wave velocity distributions v and $\tilde{v}$. Suppose they coincide, so that along the common front the velocity $\tilde{v} = v\ F(C)$ is a constant multiple of the velocity v. Their directions are identical at any given point of the common front, since both are normal to it. The sound excitation will travel in this direction, and it will travel the same infinitesimal orthogonal distance, ds, from a given point, if the constant time differentials dt and $\frac{1}{F(C)}$ dt are allocated to the respective motions. Hence, two adjacent fronts will coincide, and so will the wave front patterns in toto, if the process is continued. The argument is clinched by the observation that there is always one common front (of enclosed area zero) at the sound source.

However, it would break down if the function $\chi = C$ where to be replaced by a function not constant on the wave fronts C, since the term "adjacent" implies constant infinitesimal travel time. The significance of the integral χ can thus be understood by the physical process itself.

From the foregoing theorem it seems reasonable to start more detailed investigations with relatively simple velocity distributions. If, e.g., v is taken as dependent on height alone:

$$v = v(\eta), \tag{21}$$

the second and third of the characteristic equations (13) may be combined into a differential equation with separated variables:

$$\frac{ds}{d\eta} = - \frac{s^2 + 1}{vs} \frac{dv}{d\eta}$$

yielding the integral

$$\psi_1 = v^2(s^2 + 1). \tag{22}$$

The necessary second integral cannot be written in an explicit form. From the first two equations (13) and from relation (22)

$$\frac{d\xi}{d\eta} = \frac{1}{s} = \pm \sqrt{\frac{v^2}{\psi_1 - v^2}} ,$$

so that

$$\psi_2 = \xi \mp \int \frac{v d\eta}{\sqrt{\psi_1 - v^2}} . \tag{23}$$

It should be kept in mind that the upper sign here is true if $s > 0$, and vice versa.

A second integral may sometimes be obtained with less effort when the first and last of the characteristic equations are used:

$$\frac{ds}{d\xi} = - (s^2 + 1) \frac{v_\eta}{v} . \tag{23A}$$

This approach is preferable when, on consulting the integral (22), the quotient $\frac{v_\eta}{v}$ as it is given by the prescribed velocity field appears as a simple relation in ψ_1 and s.

Two cases in which the integral in relation (23) or (23A) is easily evaluated will be treated in detail, namely $v = \eta$ and $v = \frac{1}{\sqrt{\eta}}$. In general, all distributions in which $\frac{v_\eta}{v}$ is a rational function of v offer no difficulties; the quadrature can then be carried out with elementary functions. Another instance when this can be done is the distribution $v = e^{1 - \eta}$ which is taken up in Section VIII.

SECTION IV. THE VELOCITY DISTRIBUTION $v = \eta$ AND DISTRIBUTIONS DERIVED FROM IT

The case $v = \eta$ has been discussed in reference 1. It will be approached here from the standpoint of the general theory set forth above, and the treatment will be extended to distributions $\tilde{v} = \eta F(X)$ which had not been considered in reference 1.

The integral (23) yields

$$\psi_2 = \xi \pm \sqrt{\psi_1 - \eta^2}.$$

Since, by (22),

$$\psi_1 = \eta^2 (s^2 + 1)$$

it follows that

$$\psi_2 = \xi \pm \sqrt{\eta^2 s^2}.$$

Non-negative values of η are admitted only, since $v \geqq 0$. Therefore

$$\psi_2 = \xi + \eta s$$

where the double sign is taken care of by s.

The functions ψ_1 and ψ_2 are independent of each other. At $\xi = 0$, $\eta = 1$

$$\tilde{\psi}_1 = \tilde{\psi}_2^2 + 1 = s^2 + 1$$

so that in relation (15) $H = \tilde{\psi}_2^2 + 1$, $f = s^2 + 1$. According to relation (14) the proper expression for s is then found from

$$\eta^2(s^2 + 1) = (\xi + \eta s)^2 + 1\,,$$

as $G \equiv H$. This yields

$$s = \frac{\eta^2 - \xi^2 - 1}{2\xi\eta} \tag{24}$$

which expression is indeterminate at $\xi = 0$, $\eta = 1$ as it ought to be.

The differential equations (16) give the families of sound rays and of wave fronts in the form

$$\xi^2 - C^* \xi + \eta^2 = 1 \tag{25}$$

(with $C^* = 2 \tan \theta_0$) and

$$C = \frac{\xi^2 + \eta^2 + 1}{2\eta}\,. \tag{26}$$

The sound rays are circles with a common point of intersection at the sound source. The wave fronts are likewise circles, all strung along the η-axis with different radii, but in such a manner that no points with $\eta < 0$ appear (Fig. 1). C is therefore a positive constant (varying from +1 to + ∞). The sound excitation cannot penetrate into the region $\eta < 0$. As an inference, no ray can reach the axis $\eta = 0$ before an infinite time has elapsed (although, mathematically, the ray circles do continue into the region below). Indeed, $v = 0$ at $\eta = 0$.

Other parts of the circles will be traced by the rays if modified velocity distributions are considered. As an example, take α = const. and

$$F = \frac{\chi + \alpha}{1 + \alpha} \quad \text{with } \chi = \frac{\xi^2 + \eta^2 + 1}{2\eta} \tag{27}$$

so that

$$\tilde{v} = \frac{\xi^2 + (\eta + \alpha)^2 - (\alpha^2 - 1)}{2(1 + \alpha)} = \eta + \frac{\xi^2 + (\eta - 1)^2}{2(1 + \alpha)}\,. \tag{28}$$

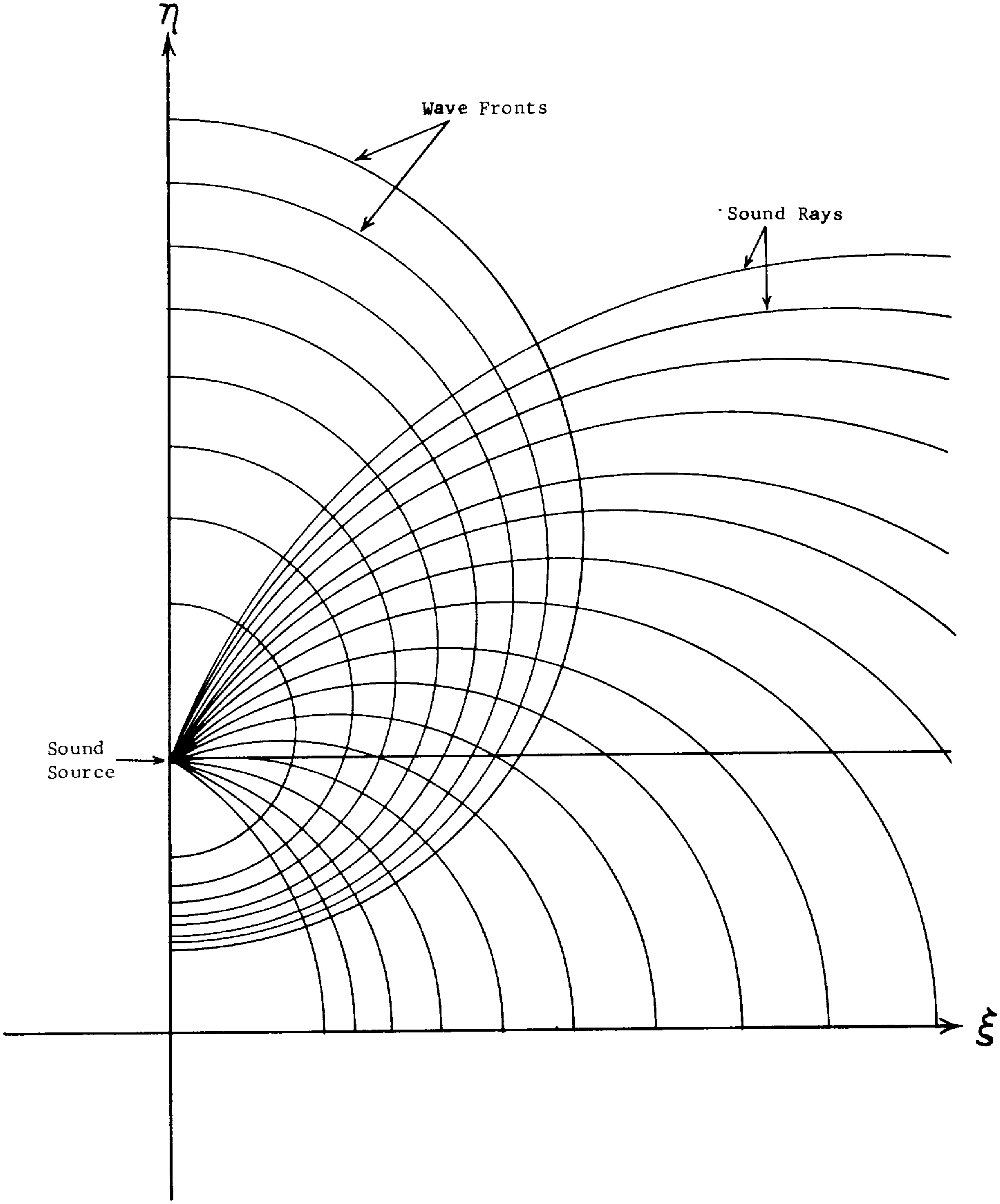

FIG. 1: WAVE FRONTS AND SOUND RAYS WITH $v = \eta$

The wave vèlocity then is constant on circles with center coordinates $\xi_m = 0$, $\eta_m = -\alpha$, whereas with $v = \eta$, it had been constant on the parallels to the ξ-axis. The propagation velocity $V_f = c^{(o)} + u_1$ cannot be negative unless the wind component is negative and its magnitude is larger than that of the thermodynamic sound speed. This will never happen near ground and, besides, is outside the scope of the present investigations which require that

$$|u_1| \ll c^{(o)}.$$

The dimensionless velocity $\tilde{v} = \tilde{V}_f/\tilde{V}_o$ is therefore not negative so that from the possible values for α the region $\alpha \leqq -1$ is preferably (though not necessarily) excluded. As also shown by expression (28), the ordinate η now may assume negative values, provided that ξ^2 is sufficiently large. There will be physically meaningful points below the ξ-axis, which in the case $v = \eta$, could not be crossed by rays (Fig. 2a). They belong to wave fronts characterized by $C < -1$.

It is obvious from expression (28) that the region $\alpha > -1$ separates into two sub-regions $|\alpha| < 1$ and $\alpha > 1$. The latter includes the case $v = \eta$, if $\alpha = \infty$. For very large values of α the velocity is constant on very flat circular arcs.

A. $\alpha > 1$

This case, especially with a rather large value of α, is probably more often nearly realized in practice than is the case $|\alpha| < 1$. The velocity $\tilde{v}$ is zero on the circles

$$\xi^2 + (\eta + \alpha)^2 = \alpha^2 - 1 \tag{29}$$

which include the degenerate circle $\eta = 0$ for $\alpha = \infty$. For any other specified value, the interior of the circle (29) is, as it were, a forbidden region; the rays arrive at the circumference no sooner than at time infinity. All the ordinates η in (29) are negative so that the bounding circle expands completely underneath the ξ-axis, its center being on the η-axis (Fig. 2a). The region outside the circle is filled with rays (unless there are material obstacles). It is now more clearly seen that the sound field extends to ordinates $\eta < 0$. The physically meaningful arcs of the ray circles are longer than in the case $v = \eta$. They end at a point of the circle (29) whose abscissa is still positive. It will be noted from equation (26) that the circle, equation (29), is the wave front characterized by $C = -\alpha$.

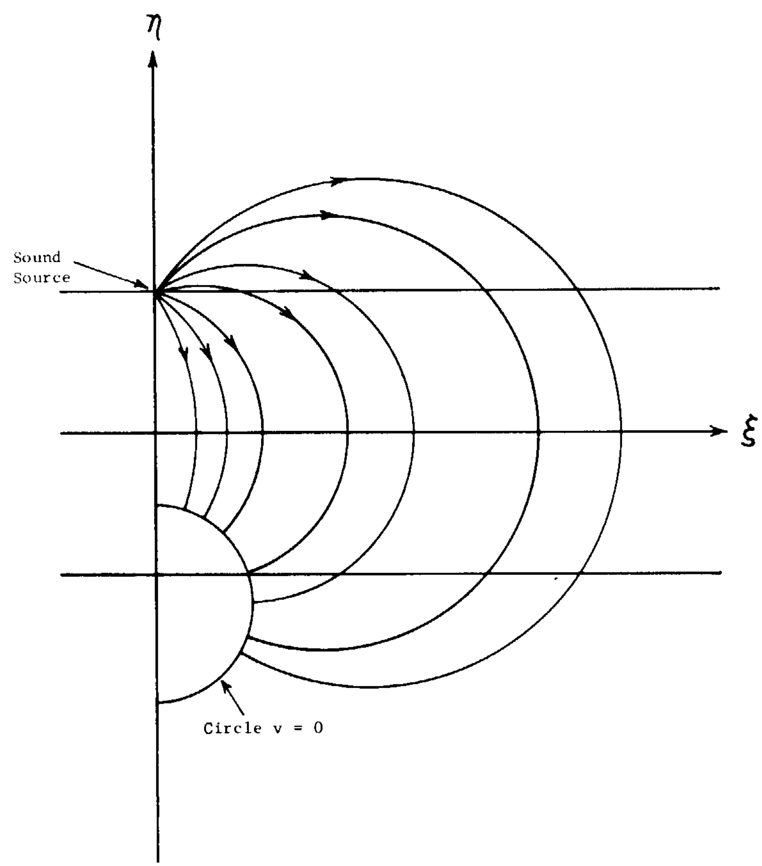

FIG. 2a: SOUND RAYS WITH THE VELOCITY, $v = \eta + \frac{\xi^2 + (\eta - 1)^2}{2(1 + \alpha)}$, $\alpha > 1$, $\lambda > 0$

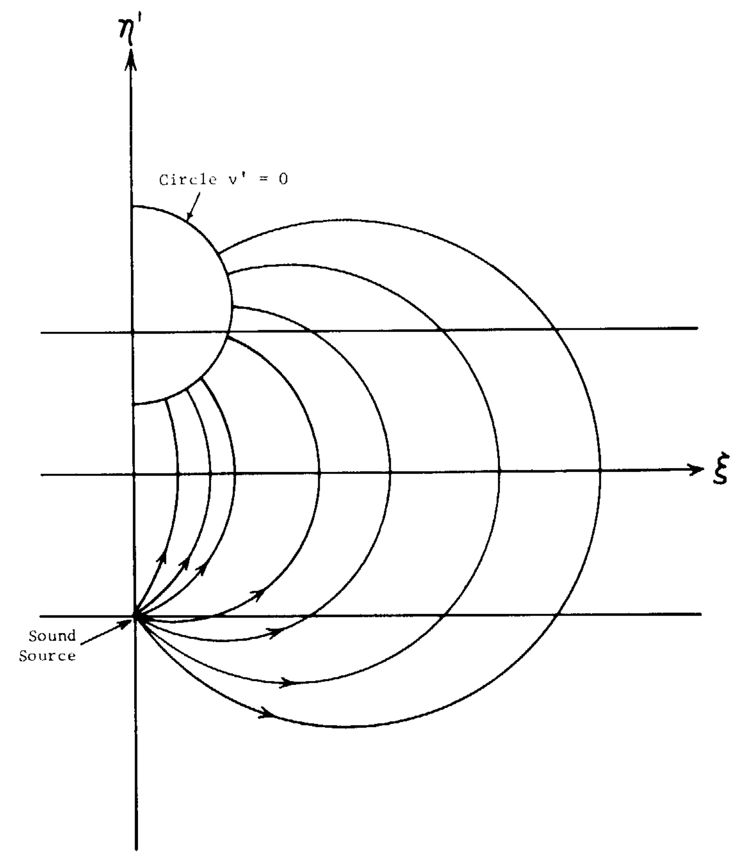

FIG. 2b: SAME VELOCITY DISTRIBUTION, SOUND RAYS FOR $\alpha > 1$, $\lambda < 0$

B. $\alpha = 1$

The circle (29) reduces to the point $\xi = 0$, $\eta = -1$. All rays converge into this mirror image of the sound source, since the dependency on C^* in equation (25) vanishes at $\xi = 0$. They arrive and end there at time infinity.

C. $|\alpha| < 1$

Since, by expression (28), the propagation velocity $\tilde{v}$ is never zero here, the motion could conceivably carry on to the left of the η-axis. However, the rays would then leave their allotted half-plane. The axis $\xi = 0$ is a natural boundary imposed by the atmospheric conditions that permit the two-dimensional treatment. The rays, arriving at the mirrored source (anti-source) with finite velocity, will continue their courses as dictated by the meteorological conditions they encounter in the opposite half-plane, that is, on curves almost always different from the circles they had been following so far. In the rare event that the state of the atmosphere is oppositely equal in opposite half-planes, the rays will circle back to the original sound source guided now by a system of wave fronts that originates at its mirror image and is pertinent to the left half-plane only, just as the first system's physical significance had been confined to the right half-plane. Thus, if a ray goes full cycle one should keep in mind that the unbroken circle it seemingly describes is actually split by the η-axis, into two arcs originating at different sources.

The foregoing qualitative description can be supported quantitatively. Let the transformations (10) be supplemented through introducing the dimensionless time variable

$$\tau = \lambda V_o t. \tag{30}$$

We wish to obtain the equations $\xi = \xi(\tau)$ and $\eta = \eta(\tau)$ for a single ray. As a preliminary step the relation $\chi = \chi(\tau)$ will be derived. The guiding idea here is to consider all the pulses leaving the source at $\tau = 0$ and to find out at which time τ they will have arrived at the wave front

$$C = \chi = \frac{\xi^2 + \eta^2 + 1}{2\eta}.$$

Since the front $\eta = 0$ is characterized by $C = \pm\infty$, a case distinction for $\eta \lessgtr 0$ becomes necessary. Above the ξ-axis the wave front constants C are positive; below it, they are negative.

With the aid of the ray slope (24) the time variation of χ along a ray, i.e., along a curved wave front normal, can be put into the form

$$\frac{d\chi}{d\tau} = \frac{\chi^2 - 1}{\xi\eta} \frac{d\xi}{d\tau}$$

$$= \frac{\chi^2 - 1}{\xi\eta} \tilde{v} \cos\theta = \frac{\chi^2 - 1}{\xi} \frac{\chi + \alpha}{1 + \alpha} \cos\theta.$$

But $\cos\theta$ is positive for $\eta > 0$ (θ in first or fourth quadrant) and negative for $\eta < 0$. Hence,

$$\cos\theta = \pm \frac{1}{\sqrt{1 + s^2}} = \pm \frac{\xi}{\sqrt{\chi^2 - 1}}$$

so that

$$\frac{d\chi}{d\tau} = \pm \frac{\chi + \alpha}{1 + \alpha} \sqrt{\chi^2 - 1}. \tag{31}$$

It is seen that, as the pulses set out from the initial (zero) wave front ($\chi = C = 1$) they will meet with fronts characterized by ever increasing values of C as long as $\eta > 0$ (upper sign); this will remain true after η has become negative and the value of C has switched from $+\infty$ to $-\infty$, because the right side in equation (31) will still be positive until the value $\chi = -\alpha$ is reached. But then, $\tilde{v} = 0$, and indeed this front is the bounding circle (29) at which the courses of all rays end, if $\alpha > 1$. Since $|\chi|$ is always ≥ 1 (as can be seen from the values on the η-axis where they are smallest), such a stoppage cannot occur in a case $|\alpha| < 1$ where in fact, as was pointed out before, the bounding circle does not exist and the rays all converge into the anti-source.

Equation (31) can be integrated with the use of the substitution

$$\chi = \frac{\alpha Q - 1}{\alpha - Q}, \quad \frac{d\chi}{dQ} = \frac{\alpha^2 - 1}{(\alpha - Q)^2}. \tag{32}$$

If $\alpha > 1$, Q increases as χ does so, beginning with Q = 1 for $\chi = 1$, arriving at $Q = \alpha$ for $\chi = \pm\infty$. With larger values of Q the denominator $\alpha - Q$ becomes negative while the numerator is always positive. Hence the square root

$$\pm \sqrt{\chi^2 - 1} = \frac{\sqrt{(\alpha^2 - 1)(Q^2 - 1)}}{\alpha - Q}$$

is negative for negative values of χ and η, as it ought to be. Equation (31) then assumes the form

$$\sqrt{\frac{\alpha - 1}{\alpha + 1}}\, d\tau = \frac{dQ}{\sqrt{Q^2 - 1}}$$

and is solved by

$$Q = \cosh \beta\tau, \quad \text{where } \beta = \sqrt{\frac{\alpha - 1}{\alpha + 1}}$$

so that

$$\chi = \frac{\alpha \cosh \beta\tau - 1}{\alpha - \cosh \beta\tau} \text{ for } \alpha > 1. \tag{33}$$

Evidently, $\chi = -\alpha$ for $\tau = \infty$

In the case $|\alpha| < 1$ a similar reasoning leads to the equation

$$\sqrt{\frac{1 - \alpha}{1 + \alpha}}\, d\tau = - \frac{dQ}{\sqrt{1 - Q^2}}$$

which is solved by

$$Q = \cos \gamma\tau, \text{ where } \gamma = \sqrt{\frac{1 - \alpha}{1 + \alpha}}$$

Then

$$\chi = \frac{1 - \alpha \cos \gamma\tau}{\cos \gamma\tau - \alpha} \quad \text{for} \quad |\alpha| < 1. \tag{34}$$

Interesting values here are:

$\chi = 1$ for $\tau = 0$,

$\chi = \pm\infty$ for $\cos \gamma\tau = \alpha$,

$\chi = -1$ for $\tau = \sqrt{\dfrac{1 + \alpha}{1 - \alpha}}\, \pi$.

The last value indicates the time at which the rays reassemble at the anti-source (since $\chi = -1$ there). It is infinite with $\alpha = 1$. However, this appears as accidental, since the substitution (32) is prohibited if $\alpha = 1$. Direct integration of equation (31) here yields

$$\chi = \frac{4 + \tau^2}{4 - \tau^2} \tag{35}$$

with $\chi = 1$ at $\tau = 0$, $\chi = \pm\infty$ at $\tau = 2$, $\chi = -1$ at $\tau = \infty$. The simplicity of this result is probably connected with the fact that, with $\alpha = 1$, the circles of constant velocity are centered at the mirror image of the sound source.

The time history of a single ray is now obtainable from equations (25) and (26), as $C \equiv \chi$. If the quantity

$$\omega = \pm \sqrt{\chi^2 - 1} \tag{36}$$

is introduced, with the upper sign holding for $\eta > 0$, the solutions for ξ and η may be written as

$$\begin{cases} \xi = \dfrac{\omega \cos \theta_0}{\chi - \omega \sin \theta_0} \\[2ex] \eta = \dfrac{1}{\chi - \omega \sin \theta_0} \quad . \end{cases} \tag{37}$$

The dependency of χ on time is given in expressions (33), (34), and (35), while that of ω emerges as

$$\omega = \begin{cases} \dfrac{\sqrt{\alpha^2 - 1} \sinh \beta\tau}{\alpha - \cosh \beta\tau}, & \text{if } \alpha > 1 \\[2ex] \dfrac{\sqrt{1 - \alpha^2} \sin \gamma\tau}{\cos \gamma\tau - \alpha}, & \text{if } |\alpha| < 1 \\[2ex] \dfrac{4\tau}{4 - \tau^2}\ , & \text{if } \alpha = 1. \end{cases} \tag{38}$$

In the degenerate simple case $\alpha = \infty$ ($\tilde{v} = \eta = v$)

$$\xi = \frac{\cos \theta_0 \sinh \tau}{\cosh \tau - \sin \theta_0 \sinh \tau}\ , \ \eta = \frac{1}{\cosh \tau - \sin \theta_0 \sinh \tau}\ .$$

These latter expressions had also been obtained in reference (1), where they appear as the set (25). Relations (37) show quantitatively the effect of an α-value $\neq \infty$ on a pulse's course along its circular path. In the upper quarter-plane the pulses will now move faster since they spend only a finite time above the ξ-axis. By the second equation (37), the time, τ^*, of arrival at $\eta = 0$ is associated with $X = \infty$, or $Q = \alpha$, so that

$$\tau^* = \sqrt{\frac{\alpha + 1}{\alpha - 1}} \cosh^{-1} \alpha, \text{ if } \alpha > 1$$

$$\tau^* = \sqrt{\frac{1 + \alpha}{1 - \alpha}} \cos^{-1} \alpha, \text{ if } |\alpha| < 1$$

$$\tau^* = 2, \text{ if } \alpha = 1 \text{ (from (35)).}$$

It can be shown that τ^* keeps decreasing with decreasing values of α, so that the pulse motion grows faster and faster when one proceeds from $\alpha = \infty$ toward $\alpha = -1$. This is what one may expect from the expression chosen for $\tilde{v}$ which contains the denominator $(1 + \alpha)$. Since there is a physical limit to the propagation velocity of sound waves, small values of α will not often occur in practice. It will be recalled that the dimensionless velocity v was defined as the ratio V_f/V_o and therefore will never be far from unity in a reasonably close neighborhood of the source. Since $\eta \approx 1$ for not too large values of $|y|$ (the quantity λ being very small, say of order 10^{-5} m^{-1}), it follows from the definitions (10) and (28) that

$$\tilde{v} \approx 1 + \frac{x^2 + y^2}{2(1 + \alpha)} 10^{-10}$$

where, with the low-lying rays alone considered, the y^2-term can be neglected for sufficiently large values of x. A 2% change (≈ 6°C) in absolute temperature corresponds roughly to a 1% change in $\tilde{v}$. (See Formula (1).) Suppose this has been observed at 100 km horizontal distance from the source; then $\alpha \approx 50$. If it was recorded at 30 km, $\alpha \approx 4$; if at 10 km, $\alpha \approx -0.5$. These figures presuppose that the velocity distribution as observed at many points can be approximated by the form (28). If this is not feasible, a function F better adapted to the observational evidence must be sought. To work out a rational method for doing this must be relegated to later investigations, as must the practical aspects in general. It may be difficult to arrive even at an approximate representation of this type, since a different basic situation will evolve when the distribution $v = \eta^{-\frac{1}{2}}$ is discussed in the next section.

It will have been noted that in the foregoing ξ had been taken as a positive quantity, the families of dimensionless wave fronts and ray curves reflecting physical reality in the right half-plane only. If the sound source is sitting on flat ground, all its rays will then return to ground level ($\eta = 1$). This seems to exclude a wide realm of conditions where in fact they do not.

The physical distance x, originally a radius vector, is always positive, but, according to the transformation (10), the quantity ξ is so only with $\lambda > 0$. With $\lambda < 0$, ξ is always negative. So is the dimensionless time τ, by the transformation (30). This suggests to substitute for ξ and τ the quantities $\xi' = -\xi$ and $\tau' = -\tau$ which then are always positive. The mathematical formulations remain unchanged in essence if in addition the substitutions $\eta' = -\eta$, $v' = -v$ are made; $v' = \eta'$ is then positive for $\eta' > 0$ as $v = \eta$ had been for $\eta > 0$. However, the sound source is now located at $\xi' = 0$, $\eta' = -1$ and thus has exchanged positions with the anti-source. Hence, rays emitted with positive elevation will never return to the source level $\eta' = -1$ (Fig. 2b). In the case $v' = \eta'$ they will all end at $\eta' = 0$, so that an observer in a balloon ranging above that line will not be aware of the presence of the source. (With $\lambda > 0$ he would notice it no matter how high he soared, being equipped with supersensitive receivers.) In the substitution (27) $X = C$ will change signs so that the family of wave fronts is now numbered differently, positive constants being below the ξ'-axis, negative constants above it. With the distribution $\tilde{v}'$ active, the observer would always register sound signals unless by an unhappy chance he drifted into the bounding circle which, if existing, is now completely above the axis $\eta' = 0$ (Fig. 2b). By equation (25) any ray with angle of departure θ_0 at the original source will coincide with a ray leaving the anti-source with elevation $\theta_0' = -\theta_0$. An initially downward course is converted into an initially upward course, and vice versa.

SECTION V. VELOCITY DISTRIBUTION $v = 1/\sqrt{\eta}$

The field $v = \eta$ includes velocities not realizable physically (those near $\eta = 0$ and $\eta = \infty$). The same is true with the distribution

$$v = \frac{1}{\sqrt{\eta}} \tag{39}$$

although it might be somewhat closer to actual atmospheric conditions as $v \to 0$ if $\eta \to \infty$. Also, the point where v becomes infinite is often buried in the ground underneath the source and is then without physical meaning.

From the integral (22)

$$\psi_1 = \frac{1}{\eta}\,(s^2 + 1)$$

the quotient

$$v_\eta/v = -\frac{1}{2}\,\frac{\psi_1}{s^2 + 1},$$

so that the differential equation (23A) goes into

$$\frac{ds}{d\xi} = \frac{\psi_1}{2}$$

yielding the second integral

$$\psi_2 = s - \frac{\psi_1}{2}\,\xi.$$

At the source

$$\tilde{\psi}_1 = s^2 + 1 = \tilde{\psi}_2^{\,2} + 1.$$

The required solution for s is then found from relation (14), i.e., from

$$\frac{s^2 + 1}{\eta} = \left(s - \frac{s^2 + 1}{2\eta}\,\xi\right)^2 + 1$$

which equation resolves into

$$s = \frac{2\eta \pm \sqrt{4\eta - \xi^2}}{\xi}. \tag{40}$$

This is the slope of the sound rays answering the distribution (39). It is indefinite at the source, if the negative sign applies. On integrating,

$$\mp\sqrt{4\eta - \xi^2} = 2 + \xi \tan\theta_0 \tag{41}$$

where a ray is characterized by its angle, θ_0, of departure at the source.

These curves are the parabolas

$$(\xi + \sin 2\theta_o)^2 = 4 \cos^2 \theta_o (\eta - \cos^2 \theta_o),$$

so that the slope (40) may also be written as

$$\eta' = \frac{\xi + \sin 2\theta_o}{2 \cos^2 \theta_o}. \tag{40A}$$

There is a limiting parabola, P^*, which does not belong to the ray family and is given by the equation

$$4\eta - \xi^2 = 0. \tag{42}$$

Outside this parabola no sound is heard. On it, the ray slope is

$$s = \frac{\xi}{2}$$

and is thus identical with the slope of P^* itself. The limiting parabola is the envelope of ray curves which, after grazing it, recede into the interior from which they had arrived (Fig. 3). At the point of contact a "second branch" of a ray parabola begins which does not pass through the sound source and is bound up with the upper signs in expressions (40) and (41). The contact point's abscissa ($\xi_c = -2 \cot g\, \theta_o$) is positive with $\theta_o < 0$; hence, rays that point upward when leaving the source do not touch the limiting parabola at all and have no second branch.

The equation of the wave fronts

$$\frac{d\eta}{d\xi} = \frac{-\xi}{2\eta \pm \sqrt{4\eta - \xi^2}}$$

has the integral

$$4(1 + \eta^3) + 3(1 + \eta)\, \xi^2 \pm (4\eta - \xi^2)^{3/2} = C.$$

Second branches (upper sign) exist for $C > 4$ only. Their physically real parts begin on the η-axis and end on the parabola P^*; they are orthogonal to sound rays receding from P^* after contact. Meaningful first branches exist for all values in the range $C \geqq 0$. (a) If $C \leqq 4$ they are closed curves about the sound source whose relevant parts are

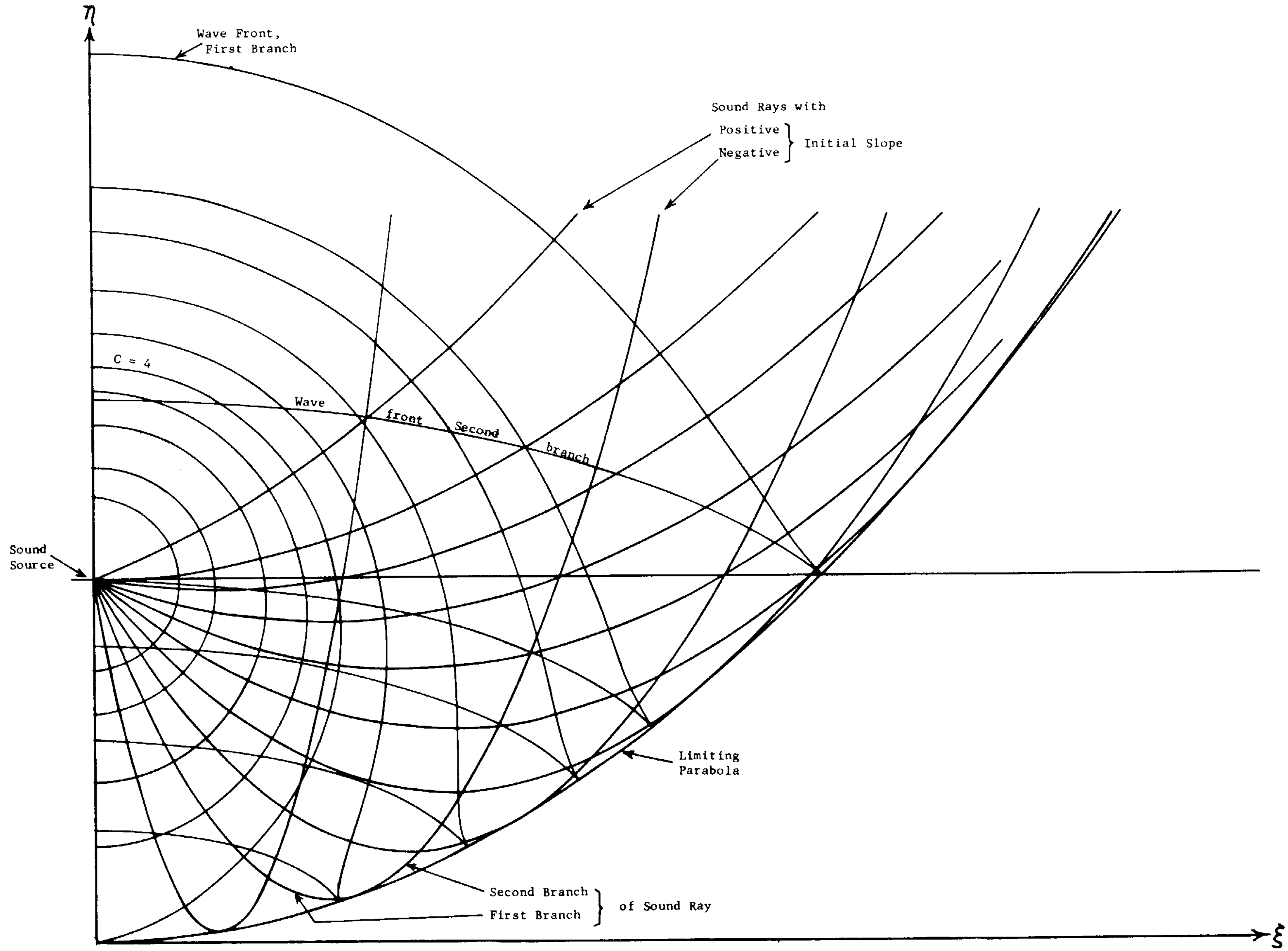

FIG. 3: WAVE FRONTS AND PARABOLIC SOUND RAYS OBTAINED FROM THE VELOCITY DISTRIBUTION $v = \frac{1}{\sqrt{\eta}}$

orthogonal to the first branches of the sound rays. (b) If $C > 4$ they stretch from the η-axis to the limiting parabola where they join the (then existing) second branch in a cusp. They complete the family of orthogonal trajectories for rays with positive initial angle of elevation.

While the wave fronts make a somewhat complex pattern (Fig. 3), the sound rays do not. Also, parabolas are among the simplest geometric curves, often more easily handled than circles. Present practical applications are mostly built on the distribution $v = \eta$ that results in circular rays. It should be worthwhile to look into the question whether the field $v = 1/\sqrt{\eta}$ where the rays are parabolas would offer advantages in the practical evaluation of layered atmospheres. At the moment it may merely be pointed out that if μ is taken as negative, i.e., if the propagation velocity

$$V_f = \frac{V_o}{\sqrt{1 + \frac{\mu}{V_o} y}}$$

increases with height, a focal point on the source level always exists, while, with the fundamental field $v = \eta$ (one-layered atmosphere), a focus cannot arise. The rays intersect with the source horizontal $\eta = 1$ at the abscissas

$$x_s = \frac{V_o}{\mu} \xi_s = - \frac{2V_o}{\mu} \sin 2\theta_o$$

so that x_s is indeed positive for $\mu < 0$ (Fig. 4).

A focal point is defined by the condition

$$\frac{dx_s}{d\theta_o} = 0$$

and is thus associated with the angle of departure

$$\theta_o = \theta_o^* = \frac{\pi}{4} ,$$

its abscissa being

$$x_f = - \frac{2V_o}{\mu} .$$

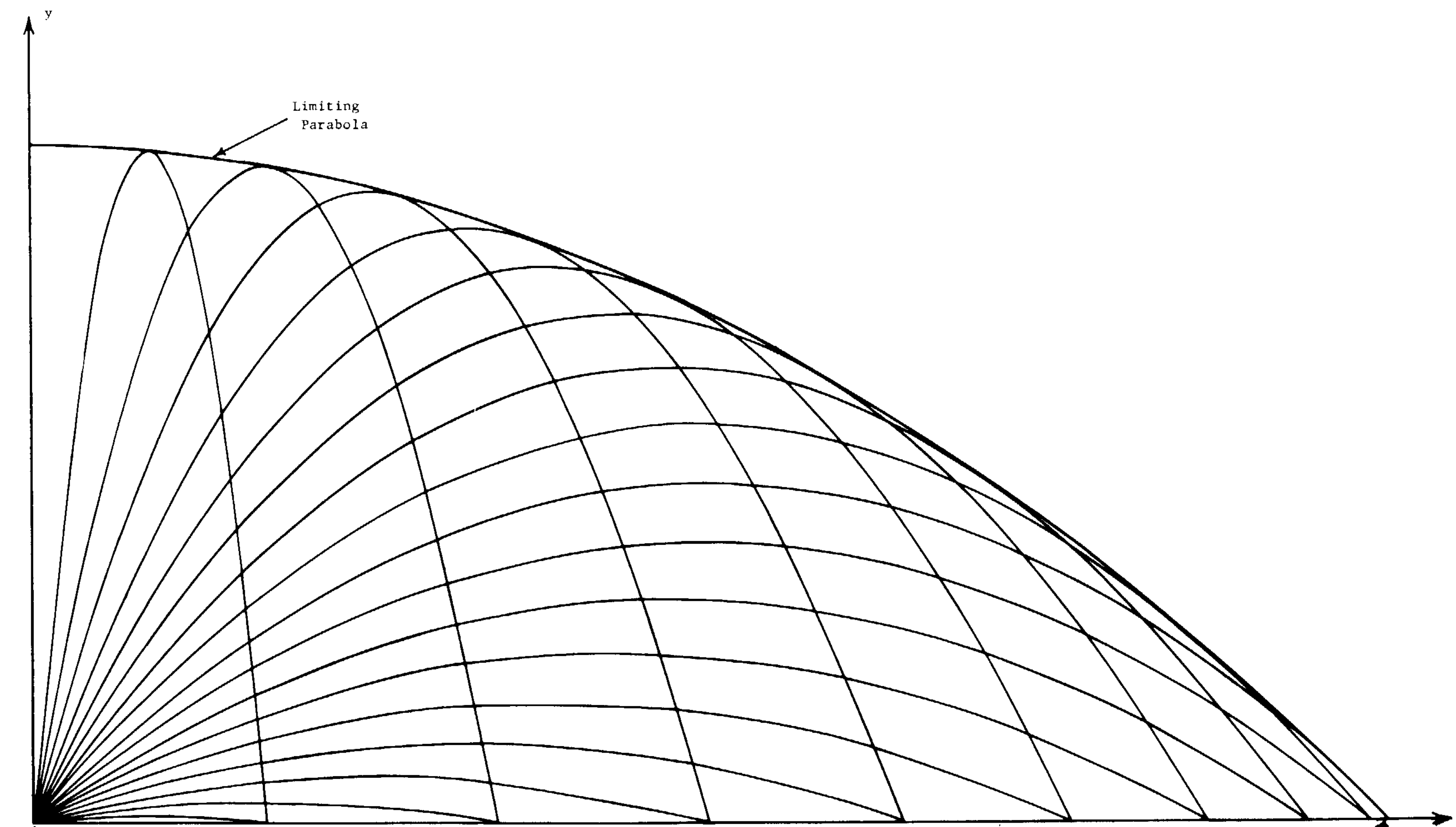

FIG. 4: PARABOLIC RAYS OBTAINED WITH THE VELOCITY DISTRIBUTION $V_F = \dfrac{V_0}{\sqrt{1 + \dfrac{\mu}{V_0} y}}$, $\mu < 0$

It will be noted that the limiting parabola

$$4\left(\frac{V_o}{\mu} + y\right) = \frac{\mu}{V_o} x^2$$

intersects with the axis $y = 0$ at the focus.

SECTION VI. HYPERBOLIC AND ELLIPTIC RAYS. THE INVERSE METHOD

Circles and parabolas depend on essentially one parameter; the two-parameter ellipses and hyperbolas can be expected to be associated with velocity fields containing one arbitrary constant more. One is, however, hard laid up in divining distributions that would send the rays on elliptic or hyperbolic trajectories. The inverse course suggests itself here: given a ray pattern, find an appropriate velocity distribution. It was shown in Section III that the distributions v and $\tilde{v} = v\,F(\chi)$ are equivalent, so that an infinite variety of such fields exists.

The basic differential equation (11) is in fact the non-dimensional planar eikonal equation and is therefore satisfied by an integral $\emptyset = \emptyset(\chi)$, as χ = const. is the equation of the wave fronts. The square of the desired velocity may then be written as

$$v^2 = \frac{f(\chi)}{\chi_\xi^{\ 2} + \chi_\eta^{\ 2}} \tag{43}$$

where the otherwise arbitrary positive function f must be chosen such that $v = 1$ at $\xi = 0$, $\eta = 1$, which is the condition for v at the source. The function χ is the integral of the wave front equation (16);

$$\frac{d\eta}{d\xi} = -\frac{1}{s}\ .$$

For simplicity take the axes of the conic section as parallel to the coordinate axes. The general equation

$$\xi^2 + 2a\xi + k\,(\eta - 1)^2 + 2b\,(\eta - 1) = 0$$

contains the coefficient k which is positive or negative for ellipses or hyperbolas, respectively. The slope, $\tan\theta_o$, at the source and the constants a and b are related by

$$a + b\tan\theta_o = 0 \tag{44}$$

so that

$$2b \tan \theta_0 = \frac{\xi^2 + k\,(\eta - 1)^2 + 2b\,(\eta - 1)}{\xi}\,.$$

The differential equation of the ray family is found by removing $\tan \theta_0$ through differentiating this equation with respect to ξ. The result contains the slope s, from which the equation of the wave fronts is found as

$$\frac{d\eta}{d\xi} = -\frac{1}{s} = \frac{2\xi\,[k(\eta - 1) + b]}{\xi^2 - k(\eta - 1)^2 - 2b(\eta - 1)}\,. \tag{45}$$

This equation is linear in the variable $u = \xi^2$; it is readily solved to give

$$\frac{\xi^2 - k(\eta - 1)^2 - 2b(\eta - 1)}{[k(\eta - 1) + b]^{1/k}} + \frac{2}{2k - 1}\,[k(\eta - 1) + b]^{\frac{2k - 1}{k}} = K. \tag{46}$$

The case

$$2k - 1 = 0$$

is not included in the general expression (46); the solution then exhibits a term $\log\,(\eta - 1 + 2b)$. With $k = 0$ the ray field is parabolic, and the formulas of Section V can be rederived on putting $b = -2\cos^2 \theta_0$; it can also be shown that the simplest possible velocity field is then $v = \eta^{-\frac{1}{2}}$. If $k = b = 1$, circular fields and the distribution $v = \eta$ arise. Taking a clue from the latter case we may simplify equation (46) by requiring that

$$b = k\ (\neq \tfrac{1}{2},\ \neq 0). \tag{47}$$

With this, the equation (46) goes into

$$\eta^{-\frac{1}{k}} \left\{\xi^2 + \frac{k}{2k - 1}\,\eta^2 + k\right\} = Kk^{\frac{1}{k}} = 2\chi. \tag{48}$$

The factor 2 is included in order to have complete agreement with expression (26) in the case where $k = 1$. The source, being a wave front of infinitesimally small enclosed area, is characterized by the constant

$$2\chi_0 = \frac{2k^2}{2k - 1}\,. \tag{49}$$

Wave fronts close to it are always associated with larger constants, as can be seen by a series expansion in terms of $\eta = 1 + \epsilon$ and $c = 1/k$. It is found that

$$2(\chi - \chi_o) = (1 + \epsilon)^{-c} \left\{\xi^2 + \epsilon^2 + 2\epsilon^3 \frac{c - 1}{3!} + 2\epsilon^4 \frac{(c - 1)\ (c - 3)}{4!} + 0(\epsilon^5)\right\} \tag{50}$$

This difference is positive if ϵ is small enough. Likewise, the denominator in expression (43) for v^2 can be written as

$$\chi_\xi^{\ 2} + \chi_\eta^{\ 2} = \frac{1}{4} \frac{1}{\eta^{2c + 2}} [(\eta^2 - 1 - c\xi^2)^2 + 4\xi^2 \eta^2] =$$

$$= \frac{1}{\eta^{2c + 2}} \left\{\xi^2 + \epsilon^2 + \epsilon[\xi^2 (2 - c) + \epsilon^2] + \frac{c^2 \xi^4}{4} + \epsilon^2(\frac{2 - c}{2} \xi^2 + \frac{\epsilon^2}{4})\right\} \tag{51}$$

where the result (48) has been used. It is then seen that if we put

$$f(\chi) = 2(\chi - \chi_o)$$

the expression

$$v^2 = 4\eta^{c + 2} \frac{\xi^2 + \frac{1}{2 - c} \eta^2 + \frac{1}{c} - \frac{2}{c(2 - c)} \eta^c}{(\eta^2 - 1 - c\xi^2)^2 + 4\xi^2 \eta^2} \tag{52}$$

takes the required value $v^2 = 1$ at the source. Since in

$$\eta = 1 + \frac{\mu}{V_o} y$$

the constant μ is of order 10^{-2} or less and since low-flung rays are only of interest ($y/V_o < 10°$), ϵ will be of order 10^{-1} or less; in these circumstances, expression (52) may be approximated by

$$v^2 \approx (1 + \epsilon)^{c + 2} \frac{\xi^2 + \epsilon^2}{\xi^2 + \epsilon^2 + \frac{c^2 \xi^4}{4} + \frac{2 - c}{2} \epsilon \xi^2 (2 + \epsilon)}. \tag{52A}$$

At the source level ($\epsilon = 0$)

$$V_f^2 = \frac{4}{4 + c^2\,\xi^2}\,V_o^2 = \frac{4}{4 + c^2\,\lambda^2\,x^2}\,V_o^2 \,.$$

Hence, if the velocity slowly decreases in x-direction the sound rays may possibly be elliptic or hyperbolic. It was shown in Section IV that if it slowly increases, a distribution of the form

$$\tilde{v} = \eta + \frac{\xi^2 + (\eta - 1)^2}{2(1 + \alpha)}$$

may be more appropriate, with which the sound rays are circles.

The inverse method can be applied to any ray pattern one cares to prescribe. It has the advantage that it requires the finding of the integral χ only, whereas the direct method calls for the solution of the system (13) with subsequent integration of equations (16). For a systematic correlation of velocity and ray fields both are equally applicable. The former may be found to have a slight edge on account of its greater simplicity. On the other hand, simple η-dependent velocity distributions are best studied with the direct method; more complicated fields depending both on η and ξ and yielding the same ray pattern can then be set up in the manner indicated in Section III. To be sure, the underlying principle applies also to velocity fields obtained by the inverse method; these, however, are apt to be complicated in themselves, as the example (52) shows, and multiplication by some function $F(\chi)$ will in general add to the complexity. One notable exemption to this rule emerges when the expression (52) is applied to circular rays ($c = 1$) where

$$v^2 = 4\eta^3\,\frac{\xi^2 + \eta^2 + 1 - 2\eta}{(\eta^2 - 1 - \xi^2)^2 + 4\xi^2\,\eta^2}$$

and, by (48)

$$\chi = \frac{\xi^2 + \eta^2 + 1}{2\eta}$$

so that

$$v^2 = \frac{2\eta^2}{\chi + 1} \,.$$

The result, on choosing $F^2(\chi) = \dfrac{\chi + 1}{2}$, simplifies into the distribution $\tilde{v} = \eta$, which is the basic distribution for circular rays.

With c = - 1 the rays are equilateral hyperbolas (Fig. 5). From expression (52)

$$v^2 = \frac{4}{3} \frac{(\eta - 1)^2 (\eta + 2) + 3\eta \xi^2}{(\eta^2 + \xi^2 - 1)^2 + 4\xi^2 \eta^2} .$$

A similar rearrangement with the aid of

$$2\chi = \eta (\xi^2 + \frac{1}{3} \eta^2 - 1)$$

is not feasible here. There is a "bounding" wave front along which $v = 0$. However, the rays are turned away before they come near it; it is therefore not indicated on Figure 5.

The bounding front is a true barrier for the sound propagation in the case of elliptic rays. Take c = 3 as an example (Fig. 6). Then

$$\chi = \frac{\xi^2 - \eta^2 + 1/3}{2\eta^3}$$

and

$$v^2 = \frac{4}{3} \eta^5 \frac{(\eta - 1)^2 (2\eta + 1) + 3\xi^2}{(\eta^2 - 1 - 3\xi^2)^2 + 4\xi^2 \eta^2} .$$

Again, a simpler velocity field bringing forth these rays is not likely to exist. The bounding wave front is defined by $\eta = 0$. The curve

$$(\eta - 1)^2 (2\eta + 1) + 3\xi^2 = 0$$

where $v = 0$, too, lies below the line $\eta = - \frac{1}{2}$ and is without physical meaning, since the rays cannot proceed beyond the line $\eta = 0$.

The restriction (47) (b = k) can be removed. Also, it is not necessary to take the quantities b and k as constants; they may depend on θ_o. In the latter case, the relations (45) and (46) are no longer applicable, and the expression for the wave front integral must be worked out according to the dependency as prescribed. The ray field may contain all types of conic sections (e.g., with $a = - \tan \theta_o$, $b = 1$, $k = \tan \theta_o$). However, novel results of a general nature have not been found in the examples so far dealt with.

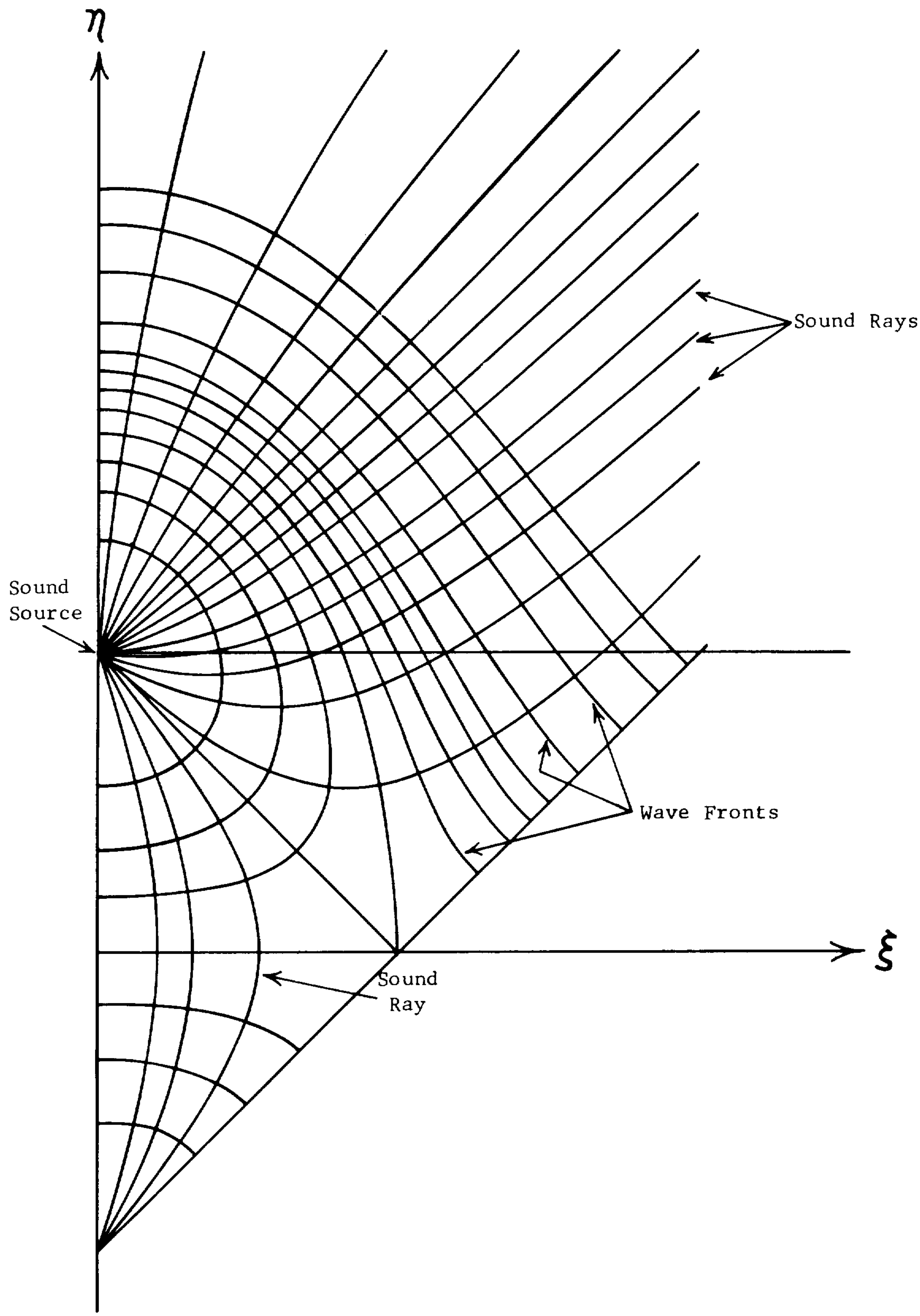

FIG. 5: HYPERBOLIC RAYS, FIRST EXAMPLE

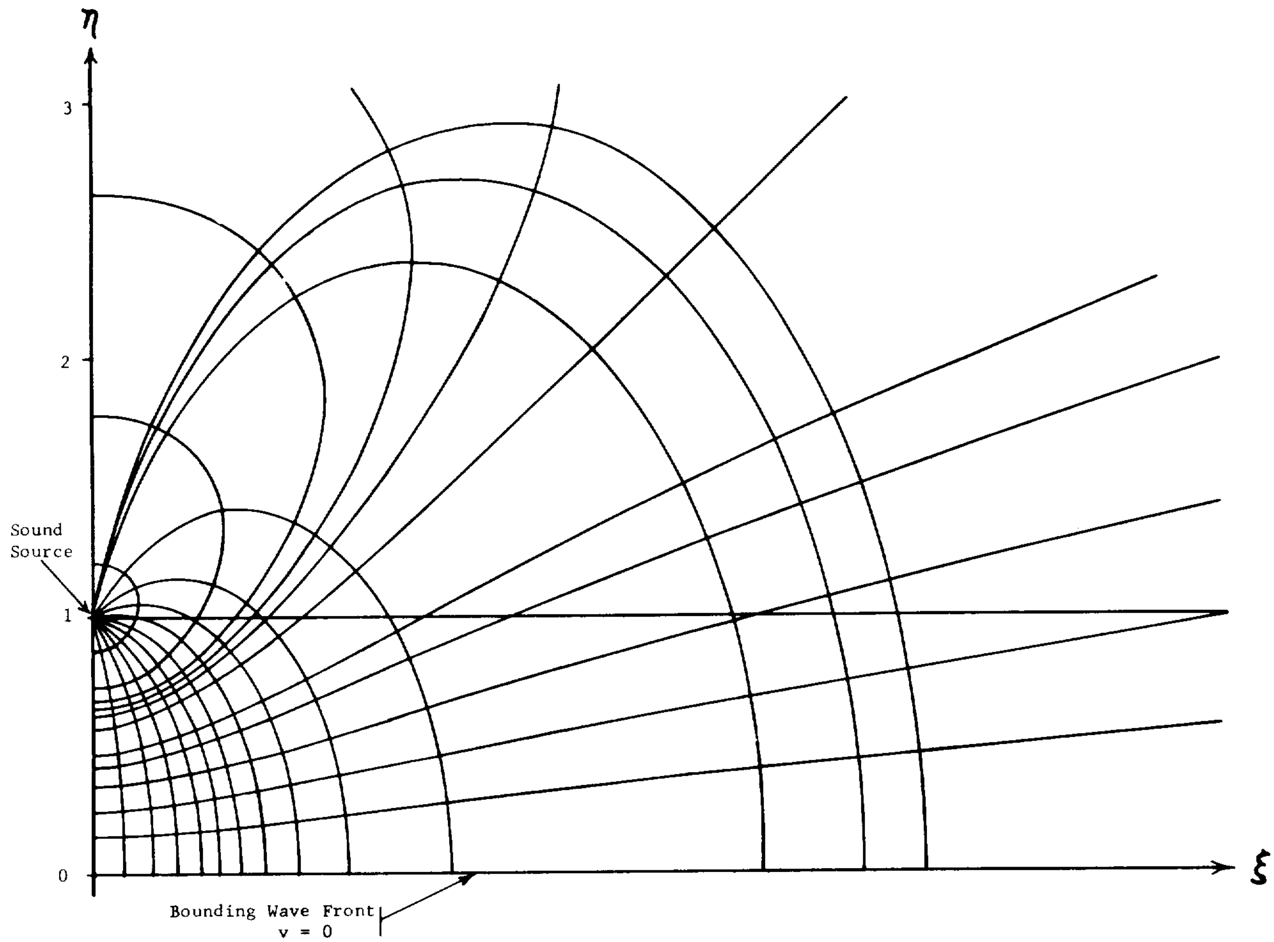

FIG. 6: EXAMPLE OF ELLIPTIC RAYS

Finally, the rays can be assumed as conic sections with axes not parallel to the axes of coordinates. A rather simple velocity field arises if the ray equation is taken as

$$\xi^2 - \eta^2 + 1 + 2\xi\eta \tan \theta_0 = 0. \tag{53}$$

These curves (Fig. 7) are hyperbolas whose axes are turned by $1/2\ \theta_0$ and have the common length $2\sqrt{\cos \theta_0}$. The center is always at the origin ($\xi = \eta = 0$), the asymptotic lines being given by

$$\eta \cos \theta_0 = \xi (\sin \theta_0 \pm 1).$$

On eliminating $\tan \theta_0$ from equation (53) the ray slope is obtained as

$$\frac{d\eta}{d\xi} = s = \frac{\eta}{\xi} \frac{\xi^2 + \eta^2 - 1}{\xi^2 + \eta^2 + 1} .$$

The wave front equation

$$\frac{d\eta}{d\xi} = - \frac{1}{s}$$

has the integral

$$\text{(a)} \quad \chi = (\xi^2 + \eta^2 - 1)^2 + 4\xi^2$$

which may also be put into the form

$$\text{(b)} \quad \chi = (\xi^2 + \eta^2 + 1)^2 - 4\eta^2 .$$

From (a) and (b), respectively

$$\chi_\eta = 4\eta \sqrt{\chi - 4\xi^2}$$

$$\chi_\xi = 4\xi \sqrt{\chi + 4\eta^2} ,$$

so that

$$\chi_\xi{}^2 + \chi_\eta{}^2 = 16\chi (\xi^2 + \eta^2).$$

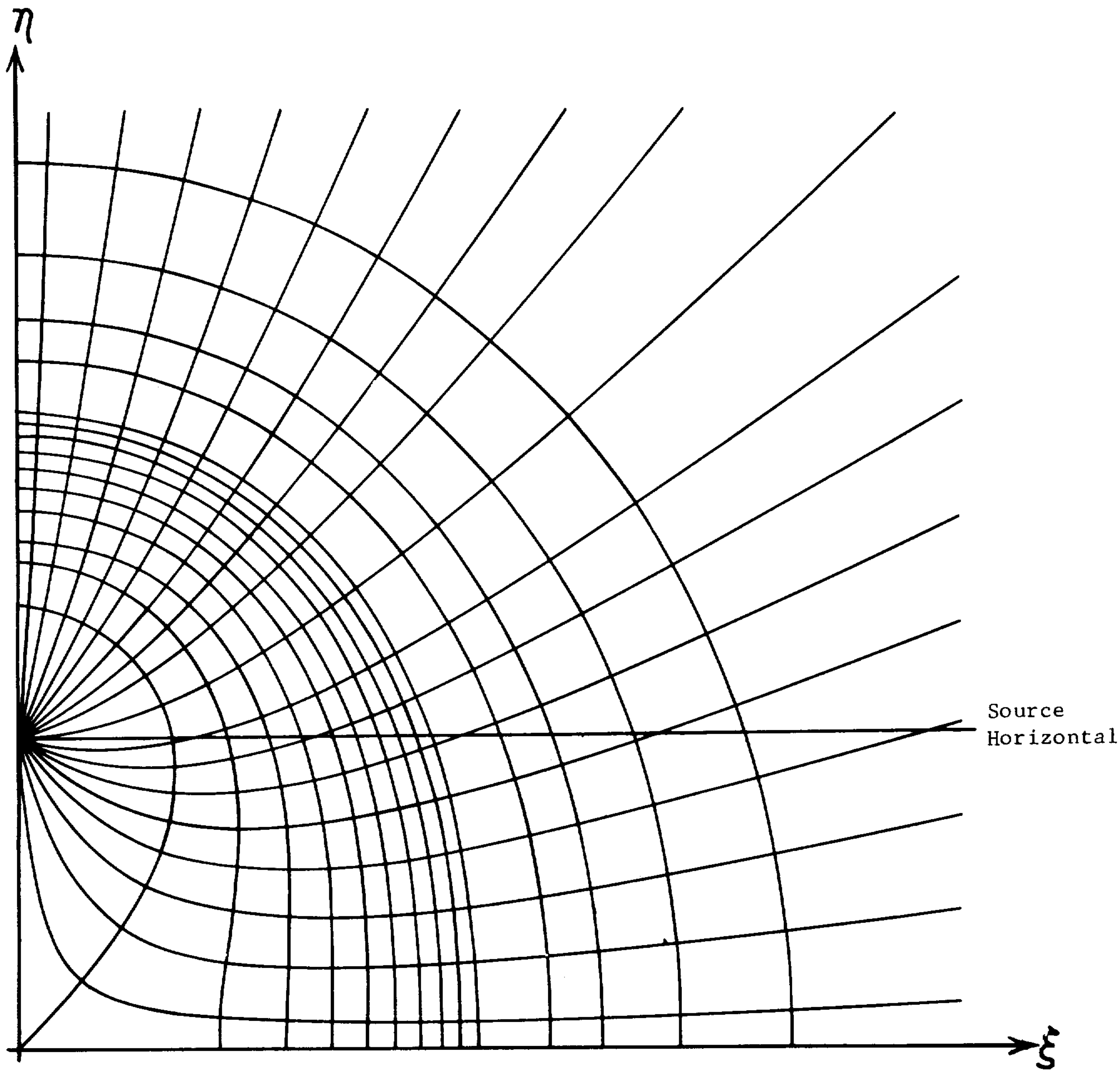

FIG. 7: HYPERBOLIC RAY FIELD, SECOND EXAMPLE

If in relation (43) the function $f(\chi)$ is taken as 16χ, the velocity field

$$v = \frac{1}{\sqrt{\xi^2 + \eta^2}} \tag{54}$$

is seen to give rise to the ray pattern (53). The transformation (10) shows that, in the physical plane, the velocity may be written as

$$V_f = \frac{V_o}{\sqrt{\lambda^2 x^2 + (1 + \lambda y)^2}}$$

and, since λ is very small, never deviates much from V_o in a reasonably close neighborhood of the sound source (as it should not, if an actual meteorological situation is to be approached by it). It is constant on the circles

$$x^2 + \left(\frac{V_o}{\mu} + y\right)^2 = \text{const.}$$

whose common center is far out on the y-axis, as V_o/μ is a very large ratio in terms of tens of kilometers. Near the sound source the velocity associated with the ray field (53) is therefore constant on circular arcs that are almost straight lines parallel to the x-axis. There is no bounding wave front, as v becomes zero only with $\xi \to \infty$ or $\eta \to \infty$.

SECTION VII. ROTATION OF THE COORDINATE SYSTEM

It has been shown in Section III that, once the ray problem is solved for a given velocity distribution v, it is also solved for all velocity fields that can be written as

$$\tilde{v} = v\ F[\chi(\xi, \eta)]$$

where χ is the original wave front integral. This generalization issues from the fact that the equation (12) is satisfied by the same slope function s if v is replaced by $\tilde{v}$. The ray and wave front patterns do not change and are described by their original equations.

Rotation of the coordinate system offers a different method of extending solutions. This time it is the eikonal equation (11) which furnishes the key for the proposed method. Its gist is to transplant by rotation a solution given in a (ξ', η')-system into the (ξ, η)-system which is linked by the transformation (10) to the physical (x, y)-system. It is clear that the functions $v(\xi', \eta')$, $s(\xi', \eta')$, $\chi(\xi', \eta')$ will go into different functions $v(\xi, \eta)$, $s(\xi, \eta)$, $\chi(\xi, \eta)$, so that indeed the ray and wave front patterns in a velocity field different from the original field are known. For brevity, let us write $v(\xi', \eta') = v'$, etc.

From the foregoing, solutions to the eikonal equation

$$\left(\frac{\partial \phi}{\partial \xi'}\right)^2 + \left(\frac{\partial \phi}{\partial \eta'}\right)^2 = \frac{1}{v'^2} \tag{55}$$

are already known for a great number of velocity fields $v(\xi', \eta')$ which satisfy the condition $v' = 1$ for $\xi' = 0$, $\eta' = 1$. The same requirement has to be met within the (ξ, η)-system, so that the appropriate coordinate transformation will be written as

$$\begin{cases} \xi' = \alpha\xi - \beta(\eta - 1) \\ \eta' - 1 = \beta\xi + \alpha(\eta - 1) \end{cases} \tag{56A}$$

with

$$\alpha^2 + \beta^2 = 1. \tag{56B}$$

The rotation thus is carried out about the point (0, 1), which preserves its location; in precise terms, the transformation couples a translation of the origin onto a rotation. The velocity goes into

$$v'[\alpha\xi - \beta(\eta - 1), 1 + \beta\xi + \alpha(\eta - 1)] \equiv v(\xi, \eta) \tag{57}$$

while the left side of equation (55), being the squared length of a gradient, must remain the squared length of the function's ϕ gradient in the (ξ, η)-system, since a mere translation and rotation does not alter the length unit. Hence, the significant result is obtained that, if the transformation (56) is carried out, the equation (55) preserves its form:

$$\left(\frac{\partial \phi}{\partial \xi}\right)^2 + \left(\frac{\partial \phi}{\partial \eta}\right)^2 = \frac{1}{v^2} \tag{58}$$

that is, it is again an eikonal equation, where now $v(\xi, \eta)$ is the function (57). The ray field associated with equation (58) can be obtained in the following way.

Suppose that the wave front integral associated with the velocity field v' is known to be

$$\chi(\xi', \eta') \equiv \chi' = \text{const.}$$

Since $\emptyset$ is constant on wave fronts, equation (55) can be set into the form

$$\left(\frac{d\emptyset}{dX'}\right)^2 = E(X'). \tag{59}$$

On applying the transformation (56) one obtains

$$\left(\frac{d\emptyset}{dX}\right)^2 = E(X) \tag{60}$$

where

$$X(\xi, \eta) \equiv X'[\alpha\xi - \beta(\eta - 1), 1 + \beta\xi + \alpha(\eta - 1)].$$

Equation (60) is another representation of the eikonal equation (58), as the same process carries (55) into (58) and (59) into (60). The transformed wave front integral therefore is

$$X(\xi, \eta) = \text{const.}$$

It follows that the wave fronts in the fields v' and v have the same geometric shape but different position. The same must be true for their orthogonal trajectories, the sound rays. This means that if the trajectories in the field v' are given by $f(\xi', \eta'; C^*) = 0$, they will, in the field v, obey the equation

$$f[\alpha\xi - \beta(\eta - 1), 1 + \beta\xi + \alpha(\eta - 1), C^*] = 0. \tag{61}$$

C^* will have to be related to the angle of departure in the usual manner.

It is seen from (57) and (61) that the simple rule applies: The interdependence of a velocity and ray field is preserved if the independent variables are subjected to the same transformation (56). The integration constant C^* must not be expressed in terms of θ_0 until after the transformation has been carried through since the latter alters the angle of departure.

For example, take $v' = \eta'$ where the ray equation is

$$\xi'^2 - C^* \xi' + \eta'^2 = 1, \text{ where } C^* = 2 \tan \theta_0'.$$

In the field $v = 1 + \beta\xi + \alpha(\eta - 1)$, where the velocity is constant on inclined straight lines, the ray equation is

$$C^*[\alpha\xi - \beta(\eta - 1)] = \xi^2 + (\eta - 1)^2 + 2[\beta\xi + \alpha(\eta - 1)].$$

If this equation is differentiated with respect to ξ, evaluation of the result at $\xi = 0$, $\eta = 1$ where $d\eta/d\xi = \tan\theta_o$ gives

$$C^* = 2\,\frac{\beta + \alpha\tan\theta_o}{\alpha - \beta\tan\theta_o}\,.$$

With the use of this expression the ray equation assumes the form

$$\left(\xi - \frac{\tan\theta_o}{\alpha - \beta\tan\theta_o}\right)^2 + \left(\eta - \frac{\alpha - \beta\tan\theta_o - 1}{\alpha - \beta\tan\theta_o}\right)^2 = \frac{1}{\cos^2\theta_o\,(\alpha - \beta\tan\theta_o)^2}\,.$$

The rays are circles whose centers are aligned on the ξ'-axis. The latter's slope in the (ξ, η)-system is $-\beta/\alpha = -\tan\varphi$ where φ is the angle through which the (ξ', η')-system is rotated into the (ξ, η)-position.

Figure 8 shows the effect of a 45°-rotation on the hyperbolic field of Figure 5.

SECTION VIII. ON THE FORMATION OF FOCAL POINTS

The ray equation always contains the angle of departure, θ_o, as this is the parameter that singles a ray out of the family of all rays. A prerequisite for the emergence of a focal point at any level, y_o, is of course the existence of rays returning to that level. Their "landing" distances, x_s, can be found from the ray equation $f(x, y; \theta_o) = 0$ by writing

$$f(x_s, y_o; \theta_o) = 0. \qquad (62)$$

For the source level, $y_o = 0$ by definition. If the quantity x_s depends on θ_o in such a manner that it becomes stationary at some critical angle $\theta_o = \theta_o^*$, the rays will arrive densely packed near the point

$$x_f = x_s\,(\theta_o^*),\ y_f = y_o$$

which is then called a focus. The focal point can be determined by solving for x_s and θ_o the equation (62) combined with the extremum condition

$$\frac{dx_s}{d\theta_o} = 0 \qquad (63)$$

whose left side follows from (62). Numerical methods must be employed if the relations (62) and (63) are too involved for an analytical solution.

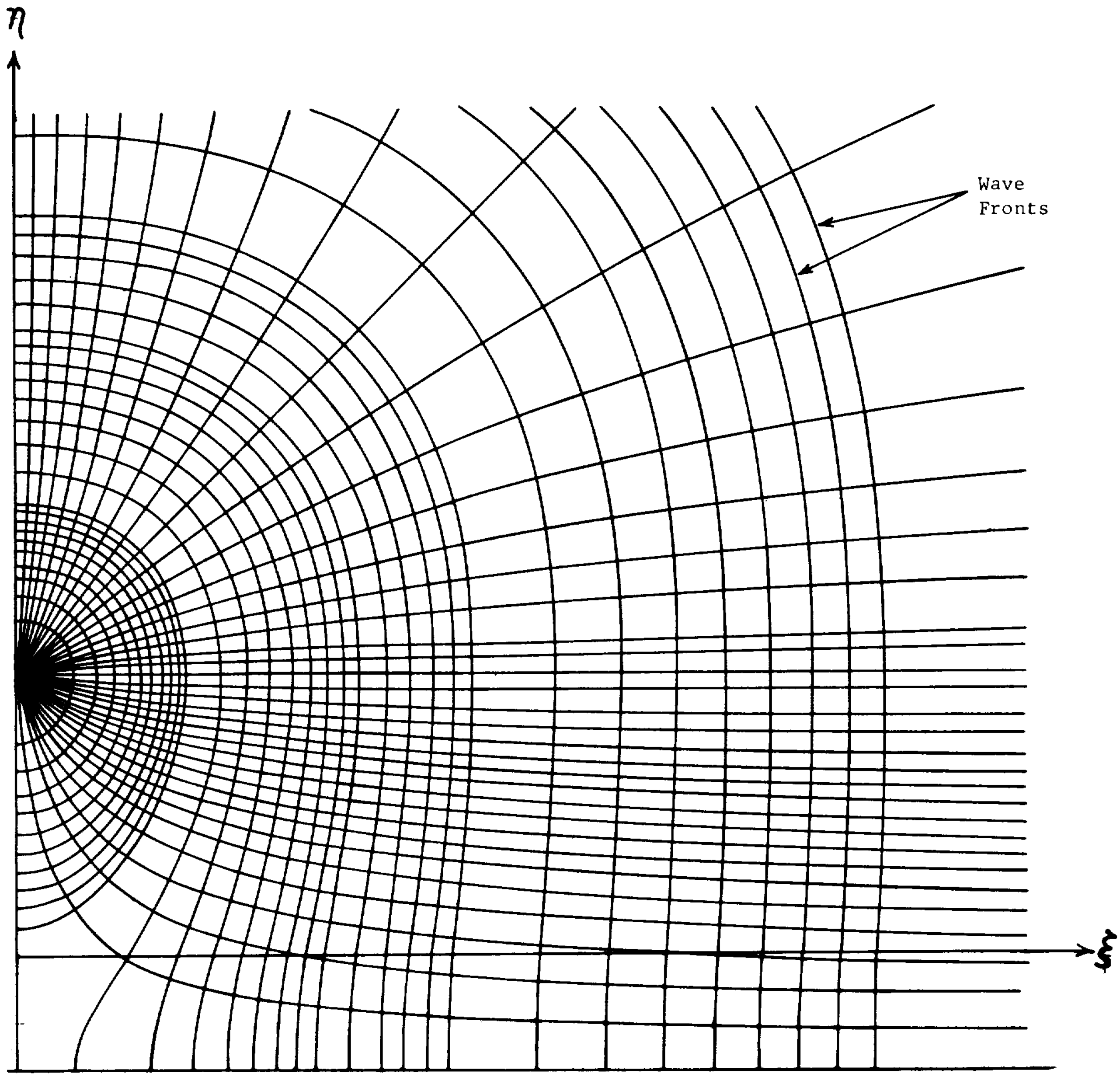

FIG. 8: HYPERBOLIC RAY FIELD; EXAMPLE FOR ROTATION METHOD

By the acoustical refractive law, returning rays are bound up with propagation velocities that increase with height at least in some portion of the atmosphere. But this is by no means a condition that would automatically result in the formation of a focal point. Positive velocity gradients merely impose on the ray a deceleration in height gain and might not succeed in turning it around completely. It can be shown that in order for this to happen the velocity must become larger than it had been at any lower level [1]. Even this is not sufficient for the existence of a focal point. A prime example is furnished by the velocity $v = \eta$ which increases linearly with height and returns all rays to the source level as long as $\theta_o > 0$. From equation (25) the dimensionless landing distances along $\eta = 1$ are

$$\xi_s = 2 \tan \theta_o$$

and thus can never become stationary with increasing θ_o.

Judging from the distribution $v = 1/\sqrt{\eta}$ one might expect that a focal point will appear if the velocity not only increases with height, but does so at an increased rate. It can be shown in an example that not even this is a sufficient condition. Take

$$v = e^{\eta - 1}. \tag{64}$$

The direct method (Section III) applies here. Since v depends on η alone,

$$\psi_1 = e^{2(\eta - 1)} (s^2 + 1)$$

$$\psi_2 = \xi \mp \int \frac{e^{\eta - 1}\, d\eta}{\sqrt{\psi_1 - e^{2(\eta - 1)}}} = \xi \mp \arcsin \frac{e^{\eta - 1}}{\sqrt{\psi_1}}$$

or

$$\frac{1}{\sqrt{s^2 + 1}} = \mp \sin (\psi_2 - \xi).$$

At the sound source ($\xi = 0$, $\eta = 1$)

$$\frac{1}{s^2 + 1} = \sin^2 \tilde{\psi}_2 = \frac{1}{\tilde{\psi}_1}.$$

According to Section III, the slope s will become indeterminate at the source, if the relationship of $\tilde{\psi}_1$ and $\tilde{\psi}_2$ is taken as the relationship of ψ_1 and ψ_2 in general. But

$$\sin \psi_2 = \sin\left(\xi \mp \text{arc sin} \frac{1}{\sqrt{s^2 + 1}}\right) =$$

$$= \sqrt{\frac{s^2}{1 + s^2}} \sin \xi \mp \frac{1}{\sqrt{s^2 + 1}} \cos \xi \ .$$

Since the square roots are taken absolute, $\sqrt{s^2} = \pm s$, depending on whether s is positive or negative. Thus

$$\sin \psi_2 = \pm \frac{s \sin \xi - \cos \xi}{\sqrt{s^2 + 1}} ,$$

and the desired value for s follows from

$$e^{2(\eta - 1)} (s \sin \xi - \cos \xi)^2 = 1.$$

The square root of the left side must be taken as (-1) (as this is required for $\xi = 0$, $\eta = 1$), so that the expression

$$s = \frac{\cos \xi - e^{1 - \eta}}{\sin \xi} = \frac{d\eta}{d\xi} \tag{65}$$

gives the slope of the sound curves in the velocity field (64). Their equation follows as

$$e^{\eta - 1} = \tan \theta_0 \sin \xi + \cos \xi = \frac{\cos (\xi - \theta_0)}{\cos \theta_0} .$$

The landing distance of rays returning to the source horizontal ($\eta = 1$) is subject to the condition

$$\cos (\xi_s - \theta_0) = \cos \theta_0$$

and is therefore

$$\xi_s = 2\theta_0.$$

Although the velocity increase with height is much more pronounced in the field $v = e^{\eta - 1}$ than in the field $v = \eta$, it still does not produce a focal point, the only effect being that the rays are returning at shorter distances. The stretch covered on $\eta = 1$ by incoming rays ends at $\xi_s = \pi$, whereas it extends to infinity with $v = \eta$. Thus, the energy return at any given spot $\xi_s \leqq \pi$ on the source horizontal will be larger than in the case of circular rays. There is no dangerous concentration of energy, however.

It seems that a general condition that would insure the existence of a focal point cannot be given either in analytic terms or in form of a rule. Every case of a velocity distribution, as things stand today, must be discussed on its own merits. It is, however, clear that focus formation is precluded if the ray family does not contain members that cross each other in the realm of physical significance. Without this the motion of the "landing" points cannot become stationary. Unfortunately, this again is not a sufficient condition, as examples have shown. Perhaps something can be achieved if ray curvatures are considered in addition to slopes; however, no effort in this direction has been made so far.

SECTION IX. CONCLUDING REMARKS

On the preceeding pages the tools have been readied for assembling an unlimited array of velocity-ray relationships in two-dimensional sound propagation. It will, however, not often occur that the actual meteorological state around the sound source will be sufficiently close to what is required for a smooth theoretical solution. To remedy this situation in the conventional case $v = \eta$, the atmosphere is divided into several horizontal layers of constant velocity gradient, and the sound propagation is determined within each layer separately. This would suggest a similar procedure in the general case $v = v(\xi, \eta)$, where the layers will not as a rule be horizontal and the boundary between layers will call for attention. Further investigations are needed to arrive at solutions in multi-layered atmospheres if the propagation velocity cannot very well be approximated by a piecewise linear expression in height alone.

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THEORETICAL ASPECTS OF PLANAR SOUND PROPAGATION IN THE ATMOSPHERE

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